The Complexity of the Game of SET

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The Game of Set - Rules

Each card has 4 attributes:

- Symbol
- Number
- Color
- Shading

The Game of Set - Rules

Each attribute can take one of 3 values:

- Symbol
  - Oval
  - Diamond
  - Squiggle

The Game of Set - Rules

Each attribute can take one of 3 values:

- Number
    - One
    - Two
    - Three

The Game of Set - Rules

Each attribute can take one of 3 values:

- Color
  - Red
  - Purple
  - Green

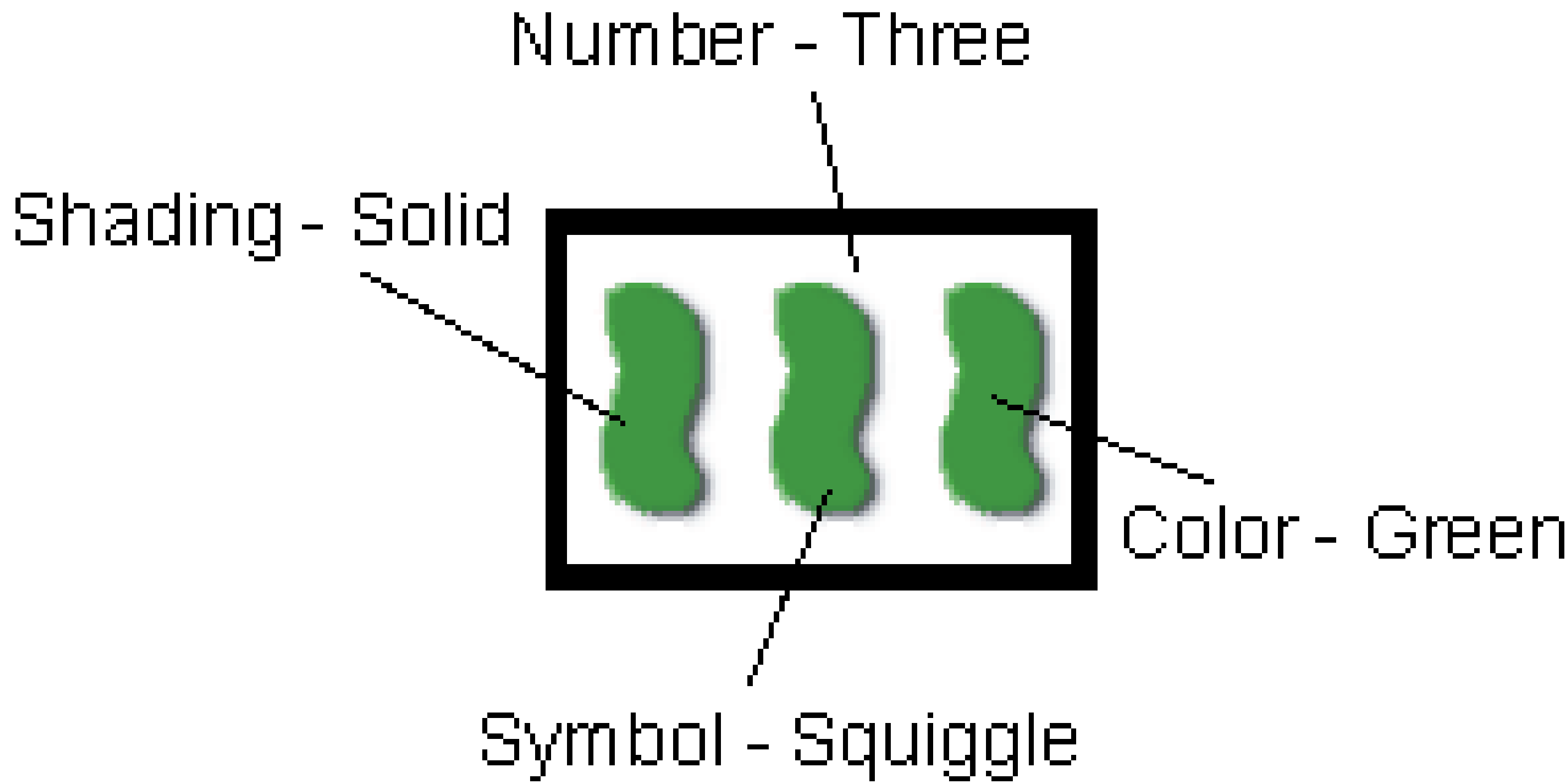
The Game of Set - Rules

Each attribute can take one of 3 values:

- Shading
  - Solid
  - Stripped
  - Blank

The Game of Set - Rules

There are $3^4 = 81$ different cards in total (one for each combination of values).

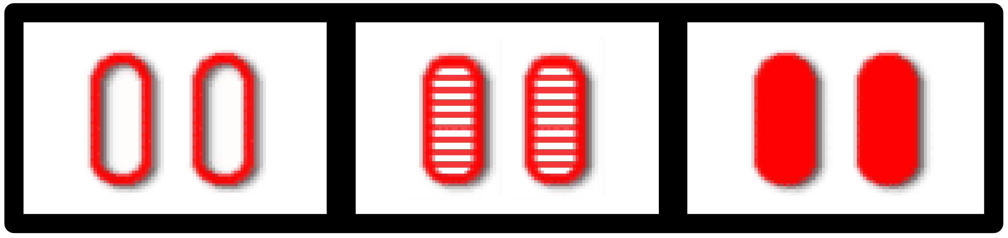


The Game of Set - Rules

- Deal 12 cards;
- Find a ***valid set****:* set of 3 cards such that, for each attribute, the values are either *all the same* or *all different.*

The Game of Set - Rules

Valid set: 3 cards with values for each attribute being either *all the same* or *all different.*



- All have **same** color;
- all have **same** symbol;
- all have **same** number;
- all have **different** shadings.

The Game of Set - Rules

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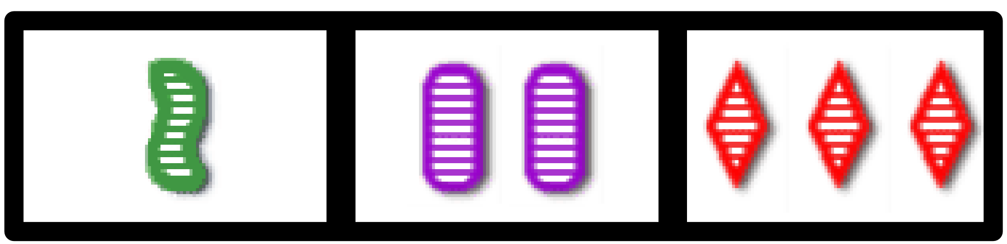
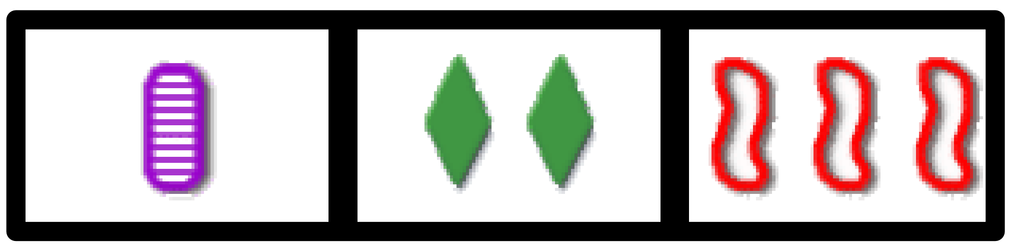


- ✓ All have **different** colors;
- ✓ all have **different** symbols;
- ✓ all have **different** numbers;
- ✓ all have **same** shading.

The Game of Set - Rules

Valid set: 3 cards with values for each attribute being either *all the same* or *all different.*



- ✓ All have **different** colors;
- ✓ all have **different** symbols;
- ✓ all have **different** number;
- ✓ all have **different** shadings.

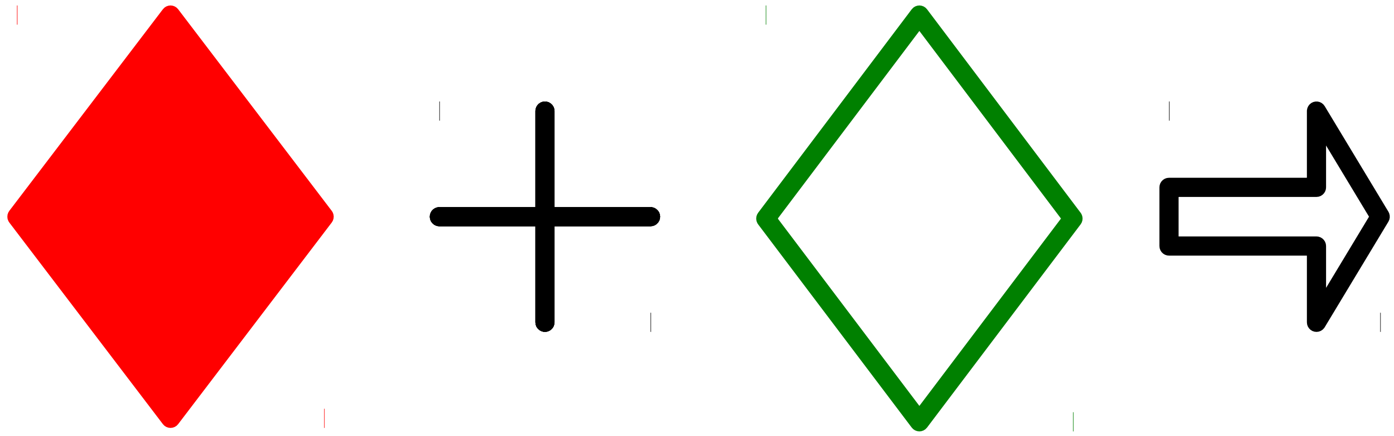
The Game of Set - Rules

This is not a valid set!

- ✓ All have **same** colors;
- ✓ all have **different** symbols;
- × **only 2/3** have same number;
- ✓ all have **different** shadings.

Interesting fact

- Each pair of cards has a unique 3rd with which they constitute a valid set.



Interesting fact

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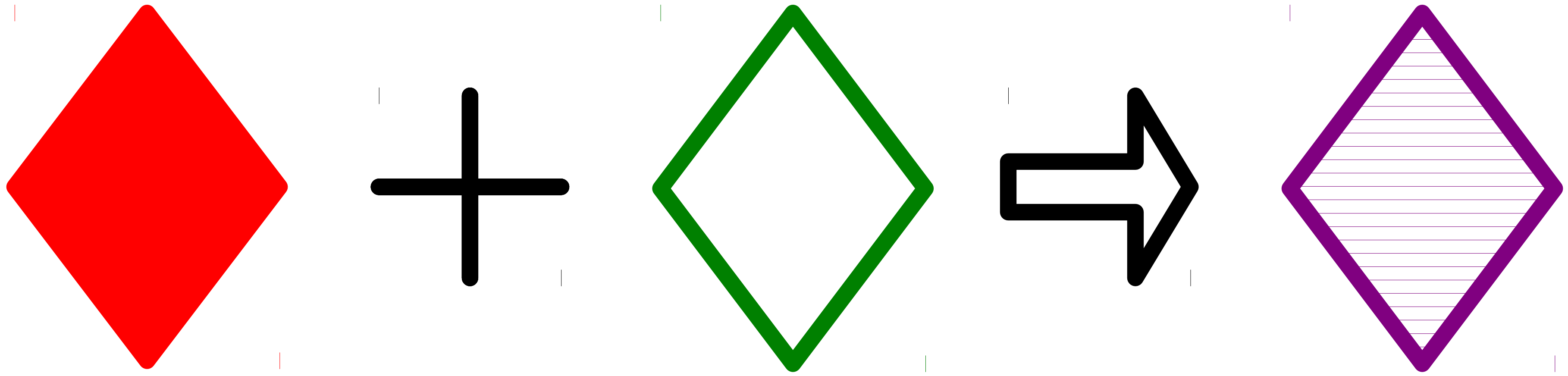
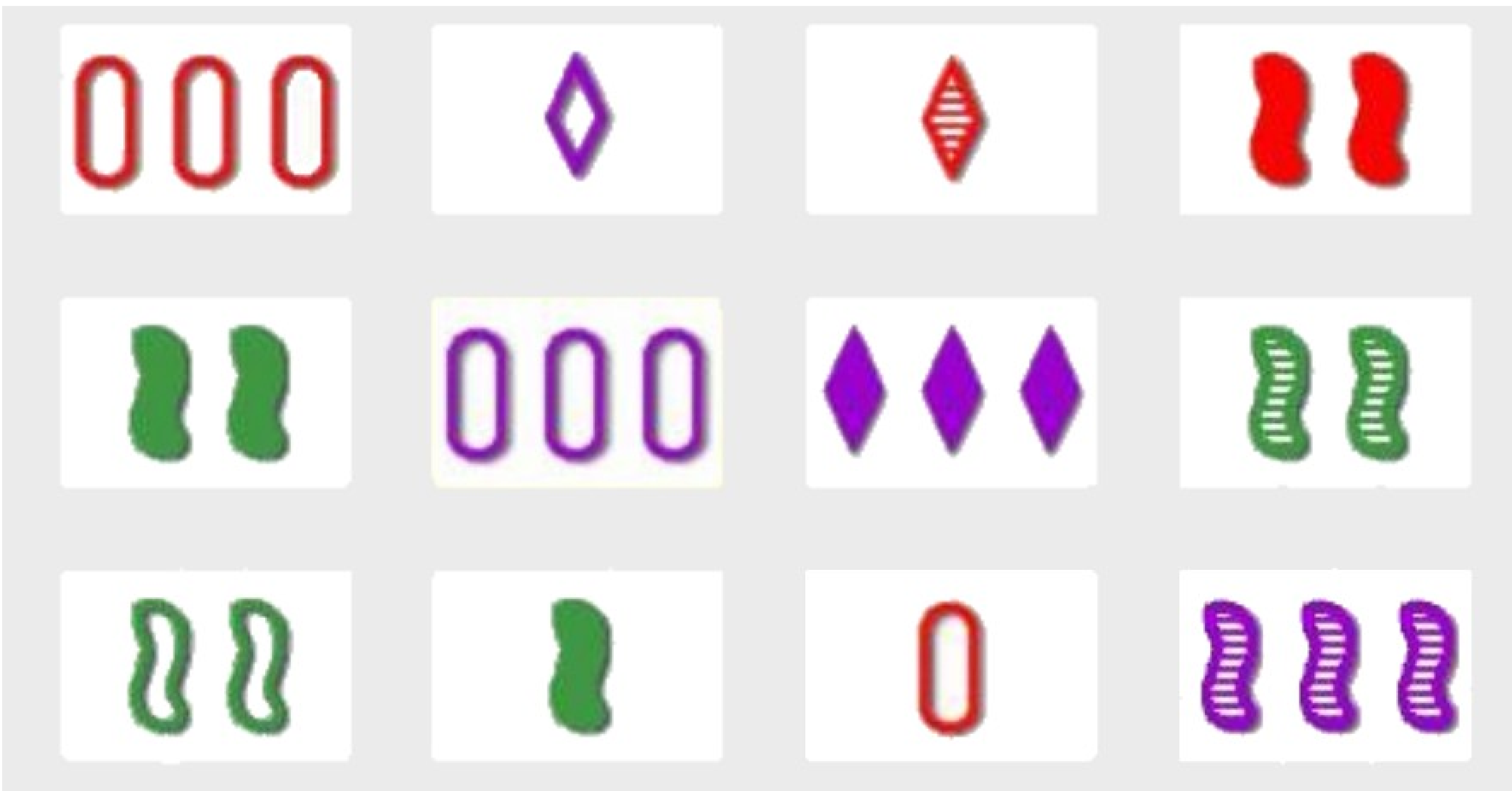


Naive way to find a set

- For each pair of cards, check whether the card that completes the valid set exists among the rest.



Generalization: k-1SET

- Input: m cards, n attributes, k values
- Question: Does there exist a valid set of k cards with all values the same or all values different?
- In the original game $m=12$, $n=4$ and $k=3$.

Complexity Results for k-1SET

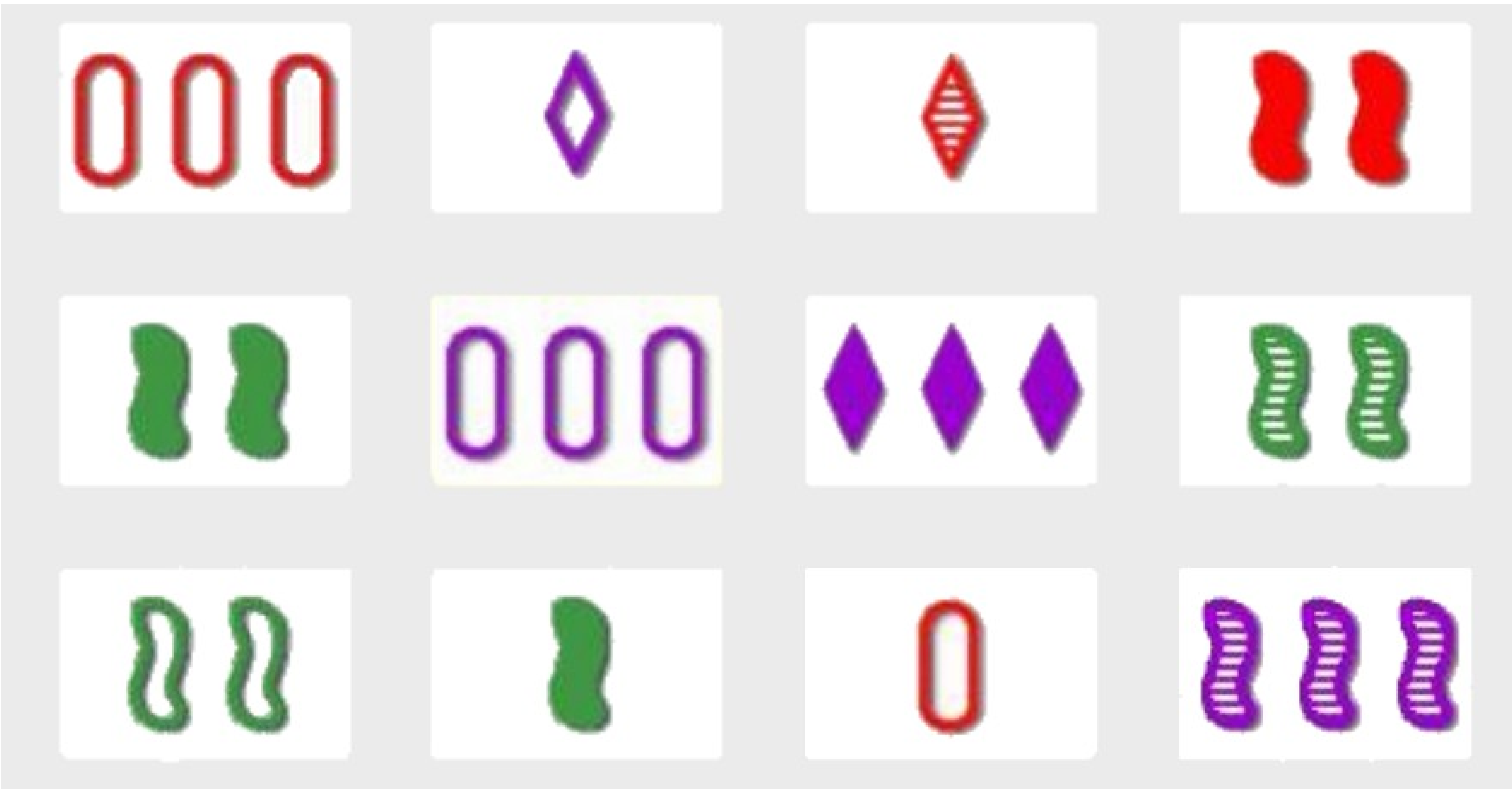
- For k unbounded:
  - $n = 2 \rightarrow$ P[1]
  - $n \geq 3 \rightarrow$ NP-Complete[1]
- For n unbounded:
  - $k = 2 \rightarrow$ trivial
  - k parameter $\rightarrow$ XP[1]

k-1SET parameterized by k is W-hard
(reduction applies to a version of *n-Dim Matching*)

1. Chaudhuri et al: *On the complexity of the game of set* (manuscript 2003).

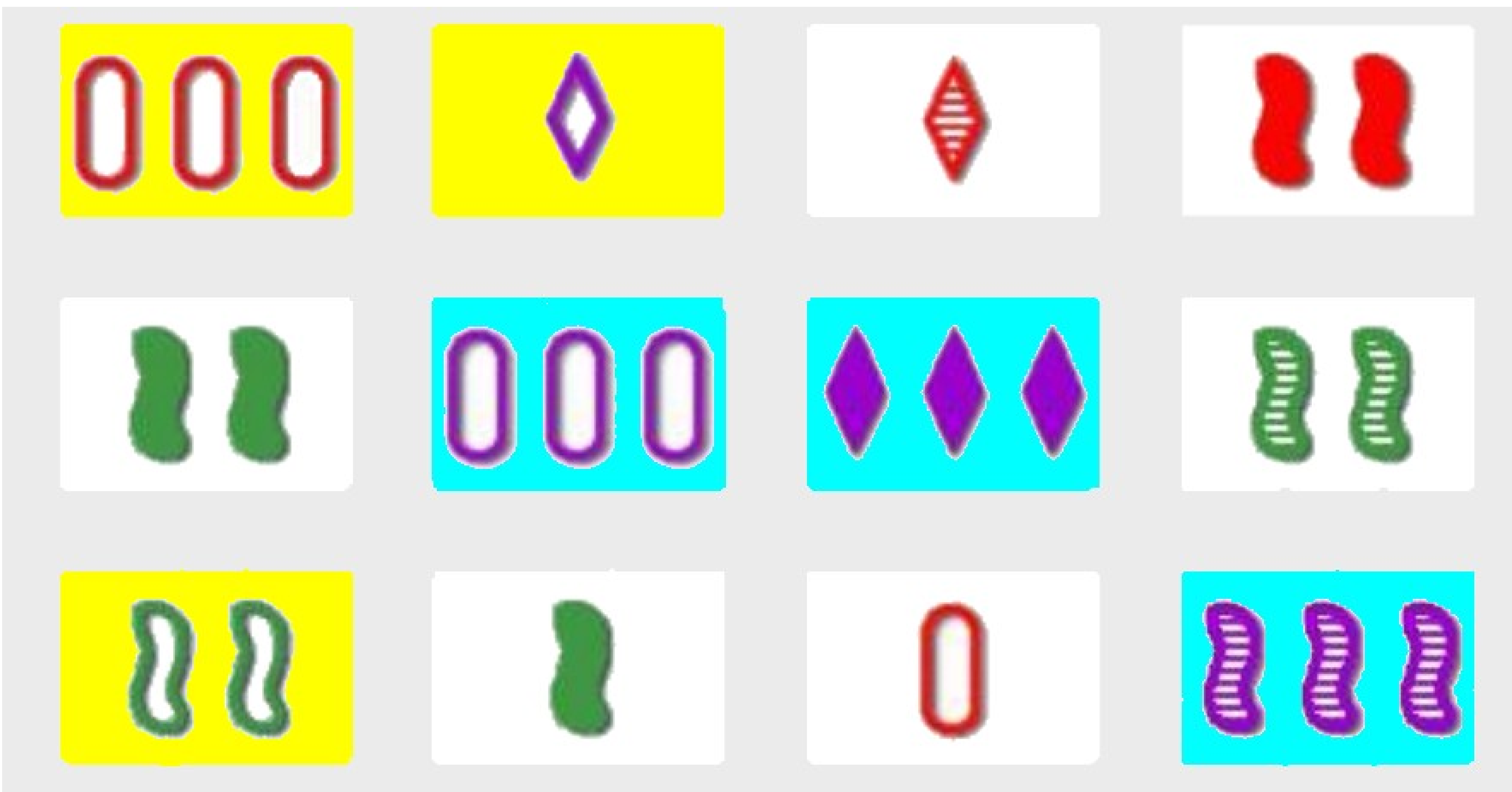
Multi-round variations

- Naive algorithm works for multi-round version, seeking the max number of co-existing valid sets.



Multi-round variations

- However, finding the max number of valid sets with card removal is not so easy (even for *k*=3).



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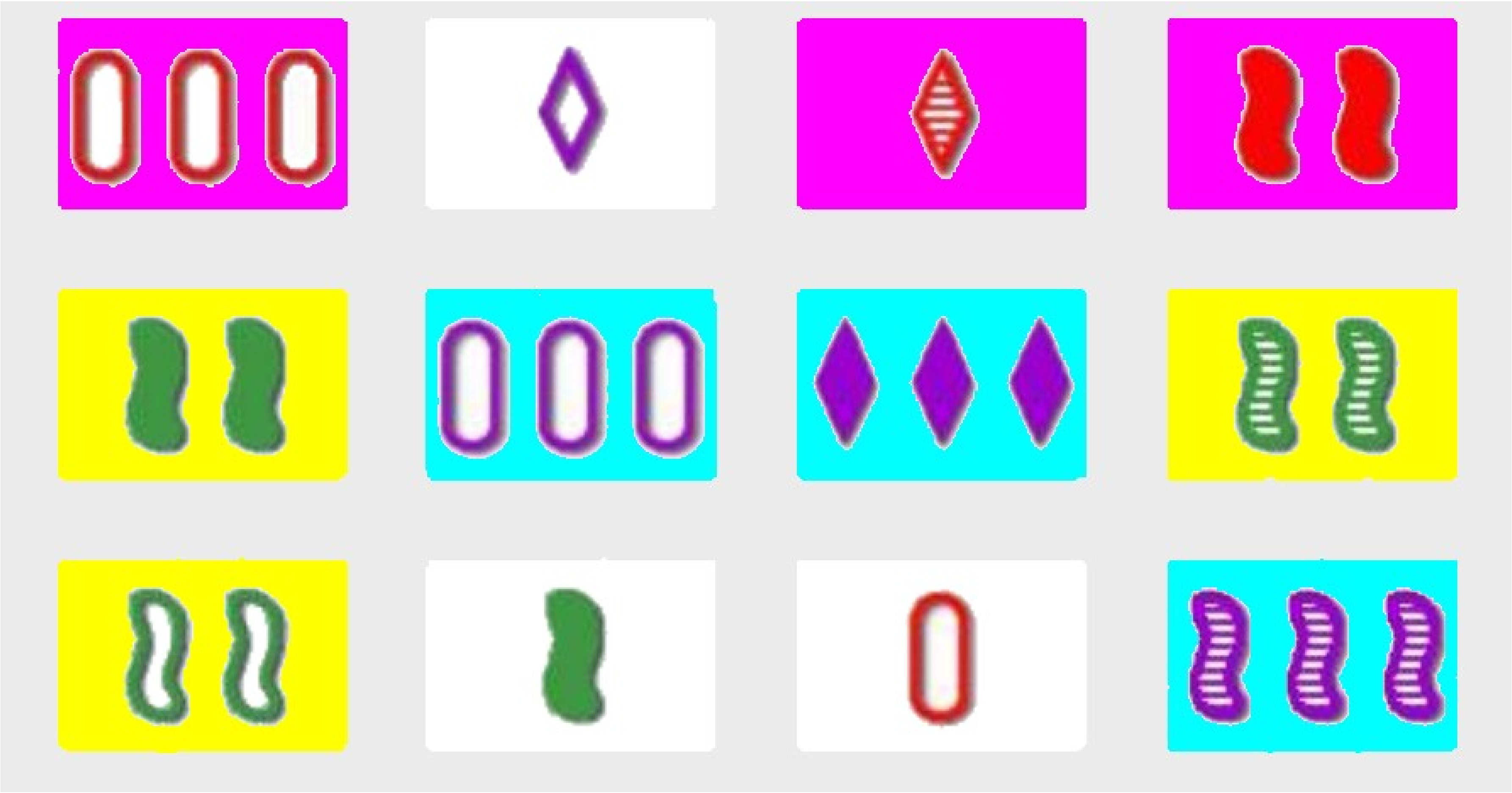


Multi-round variations

For $k = 3$ and *unbounded* n:

- Find a single valid set is in P.
- Find max # r of co-existing valid sets is in P.
- max-3-rSET: Find the max # r of existing valid sets with card removal.
- min-3-rSET: Find the min # r of valid sets that destroy all other valid sets.

Multi-round variations

Construct a 3-uniform hypergraph:
vertices ↔ cards, hyperedges ↔ valid sets.

- max-3-*r*SET: restriction of 3-Set Packing to SET hypergraphs).
- min-3-*r*SET: restriction of Independent Edge Dominating Set to SET hypergraphs).

  - *Both restrictions remain NP-hard.*
  - *min-3-rSET parameterized by r is FPT*

Interesting fact for 3-SET

Assign each value a number among {0,1,2}.
Then, each card is an *n*-tuple of ternary coordinates.

In a valid set, if we add together corresponding coordinates we always get 0 (mod 3).

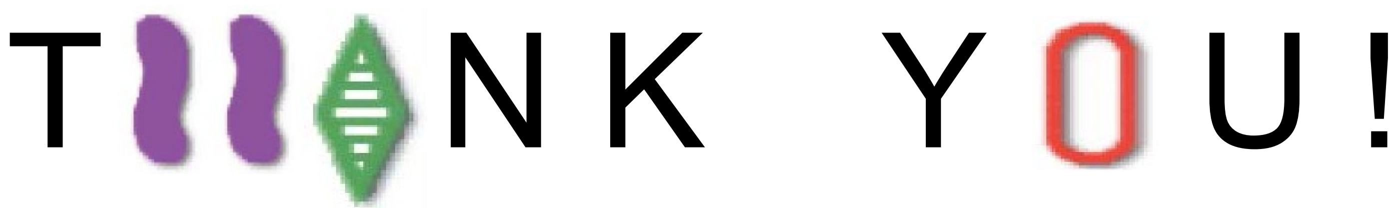
<u>Proof</u>

For a particular attribute:

- all values the same: $3*i = 0 \pmod 3$;
- all values different: $0+1+2 = 0 \pmod 3$.

Conclusions

- We studied the complexity of single- and multi-round variations of the game of SET.
- We established connections of the game with Multi-Dimensional Matching, Set Packing, Independent Edge Dominating Set.



Michael Lampis, Valia Mitsou: T**he Computational Complexity of the Game of Set and its Theoretical Applications.** *Latin 2014.*