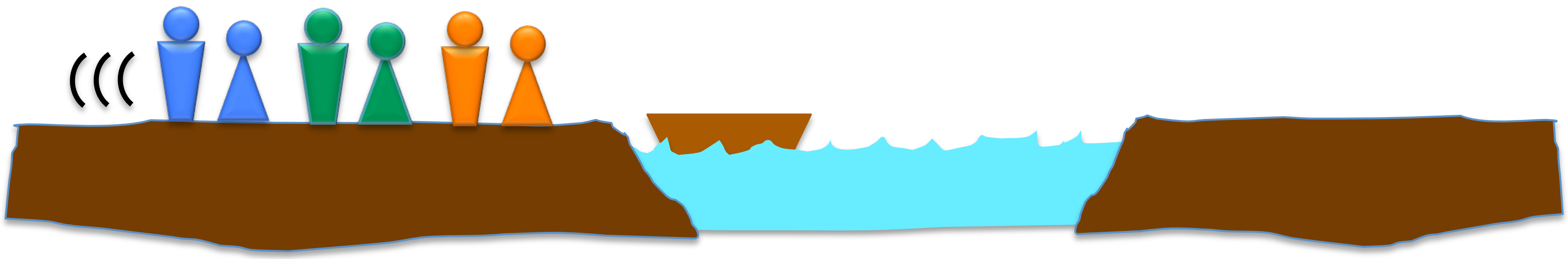
一般化川渡り問題の 多項式時間可解性

伊藤大雄 (電気通信大学)

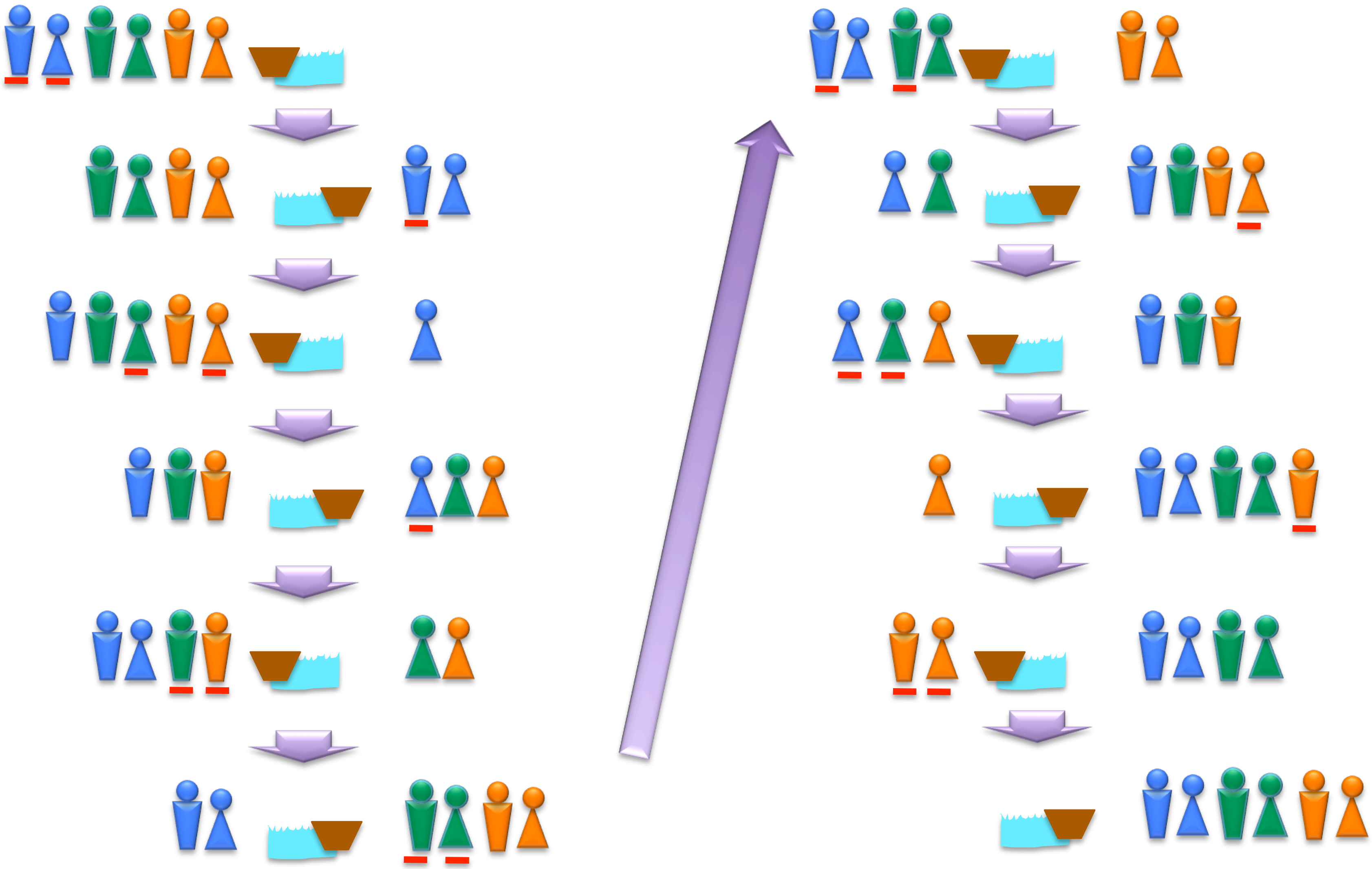
Joint work with:
Stefan LANGERMAN (Universit´e Libre de Bruxelles (ULB), Belgium)
吉田悠一 (国立情報学研究所, Preferred Infrastructure,
JST ERATO 河原林巨大グラフプロジェクト)

３人の嫉妬深い男とその妹の問題

- [Alcuin of York, 「青年達を鍛えるための諸問題」より]
- ３人の男がそれぞれ一人ずつ未婚の妹を連れて川にさしかかった。
- 二人乗りのボートが一艘あり、それを使って6人が向こう岸に渡りたい。
- ただし、男は警戒心が強く、自分の妹が、自分の居ない所で他の男と同席することは我慢できない。
- 上手く渡る方法を示せ。

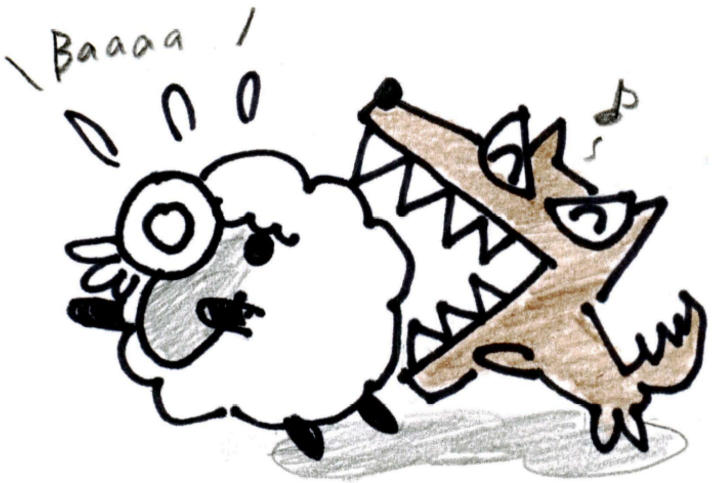


Answer



狼と山羊とキャベツの問題

- [Alcuin of York,「青年達を鍛えるための諸問題」より]
- 農夫が狼と山羊とキャベツ(の山)を持って川にさしかかった。
- ボートが一艘あり、それを使って向こう岸に渡りたい。
- ボートには、農夫+(狼か山羊かキャベツのどれか)しか一度に載せられない。
- 農夫が居ないと「狼は山羊を食べ」「山羊はキャベツを食べ」てしまう。
- 上手く渡る方法を示せ。

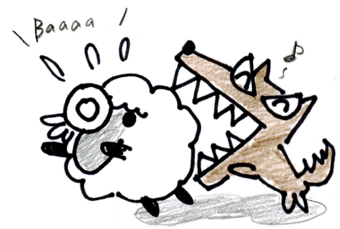


このような問題は「川渡り問題 (River Crossing Problems)」と呼ばれ、多くのバリエーションがある。

We introduce: Generalized River Crossing Problem

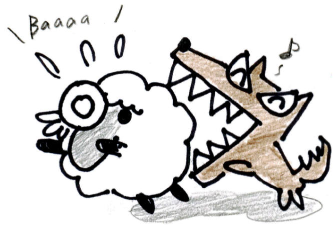
PROBLEM: RIVER CROSSING [I,Langerman,Yoshida; FUN2012]

- INSTANCE:
  - A set of drivers D, A set of customers C, Forbidden sets $\mathcal{F}_L$, $\mathcal{F}_R$, $\mathcal{F}_B \subset 2^{D \cup C}$,
  - The size of the boat $b \in Z^+$, The bound of the # of transportations $T \in Z^+ \cup \{\infty\}$.
- QUESTION: Is there a way to transport all of the drivers and customers from the left bank to the right bank of a river using a boat under the following restrictions?
- RESTRICTIONS:
  1. Initially all drivers, all customers, and the boat are on the left bank.
  2. The capacity of the boat is *b*.
  3. Only drivers can operate the boat.
  4. It is forbidden to transport exactly the members of a forbidden set in $\mathcal{F}_B$ in the boat.
  5. It is forbidden to leave exactly the members of a forbidden set in $\mathcal{F}_L$ (resp., $\mathcal{F}_R$) on the left bank (resp., the right bank).
  6. The number of transportations is at most *T*.



Ex.) the Wolf-Goat-Cabbage Problem

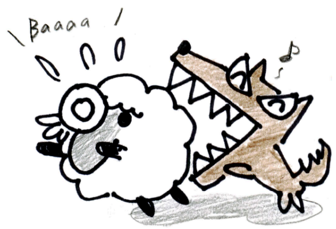
- D={Man}, C={Wolf, Goat, Cabbage}
- $\mathcal{F}_L = \mathcal{F}_R$ = {{Wolf, Goat}, {Goat, Cabbage},
- {Wolf, Goat, Cabbage}},
- $\mathcal{F}_B = \phi$,
- b = 2, T = ∞.



Note

- $T=\infty \Leftrightarrow T=2^{|P|+1}$ ($P = D \cup C$).
- In this paper, we assume that every driver is allowed to operate the boat alone (Independent-driver model).

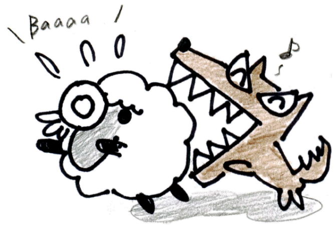
- Another formulation by using ALLOWED SETS: This can be reduced to a shortest path problem and can be solved in poly. time.



Past Work

- The allowed set formulation and shortest path approaches [B.R. Schwartz 61], [R. Bellman 62] .
- Unique driver, forbidden pairs (represented by a graph G), forbidden pairs cannot be in the same place without supervision from the unique driver; How large b should be? -> Alcuin # of G [P. Csorba, et al. ESA07] [M. Lampis, V. Mitsou FUN07].

- Our formulation is far more flexible.
- But there exists a point that makes the complexity be smaller, and it keeps this formulation interesting.



Results shown in FUN2012

- **Theorem 1**. RIVER CROSSING is NP-hard even if $\mathcal{F}_L = \mathcal{F}_R = \phi$ and b=3.
- **Theorem 2**. If $\mathcal{F}_L = \mathcal{F}_R = \phi$ and b=2, then (independent-driver model) RIVER CROSSING can be solved in polynomial time.
- **Theorem 3.** If |D|=1, b=2, T=∞, and $\mathcal{F}_B=\phi$, then RIVER CROSSING can be solved in polynomial time.

- **New Results:**
- **Theorem 4.** If T=∞, and $\mathcal{F}_B=\phi$, then RIVER CROSSING can be solved in polynomial time.

How large is the input size?

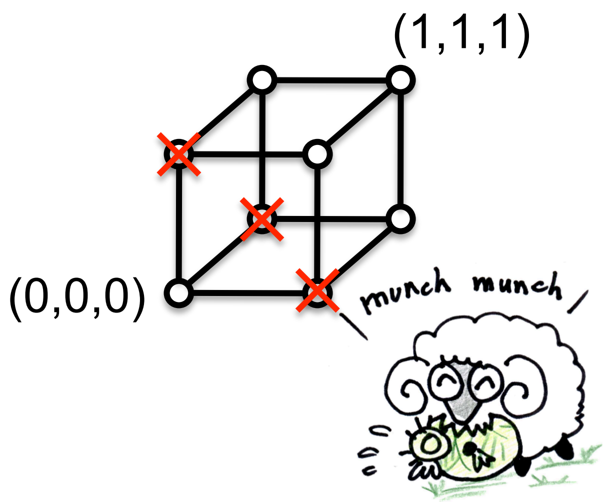
- $\mathcal{F} := \mathcal{F}_L \cup \mathcal{F}_R \cup \mathcal{F}_B$. $||\mathcal{F}|| := \Sigma_{F \in \mathcal{F}} |F|$.
- The size of input: θ(|C|+|D|+||F||)?
- But, |C| and |D| can be represented by O(log|C|) and O(log|D|) space.
- →If $||\mathcal{F}||$=o(|C|+|D|), the size of input is o(|C|+|D|+$||\mathcal{F}||$).
- Not a problem!
- Lemma 1. If $||\mathcal{F}||$=o(|C|+|D|), the answer is always ``YES'' (for T=∞).
- Proof: If $||\mathcal{F}||$=o(|C|+|D|), there is a pair p,p'∈C∪D such that
-     p,p'∈F for all F∈$\mathcal{F}$ or p,p'∉F for all F∈$\mathcal{F}$.
- i.e., any configuration separating p and p' is allowed. By using this fact, there is a simple schedule. (Detail is omitted.) □
- Therefor, we can assume $||\mathcal{F}||$=Ω(|C|+|D|).

For T=∞ (reachability prob.)

- $H_n=(V_n,E_n)$: the n-dimensional hypercube.

PROBLEM: SUB-HYPERCUBE CONNECTIVITY:

- INSTANCE: A dimension $n \in Z^+$. A set of forbidden vertices $F \subseteq V_n$.
- QUESTION: Is there a path that doesn't use any vertices in F starting from 0 = (0, . . . , 0) and destining to 1 = (1, . . . , 1)?



- Ex.) n=3, F={(0,0,1), (0,1,0), (1,0,0)}
- ⇒Answer "No."

SUB-HYPERCUBE CONNECTIVITY

- Lemma 2: There is a polynomial time algorithm for SUB-HYPERCUBE CONNECTIVITY.

- For a graph G=(V,E), S,T⊆V,
- E(S,T):={(s,t)∈E | s∈S, t∈T},
- E(S):=E(S,V-S).
- The edge expansion of S of an n-regular graph G is

$$h(S) := \frac{|E(S)|}{n \cdot \min\{|S|, |V - S|\}}.$$

- The edge expansion of G is $h(G) := \min_{S \subseteq V} h(S)$.

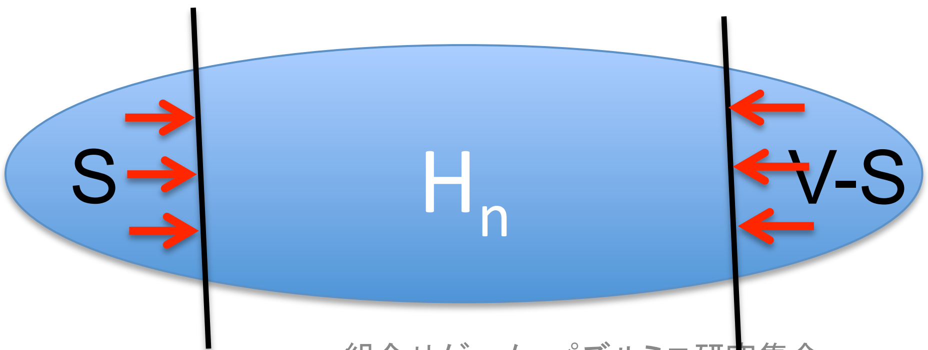
- Lemma 3: $h(H_n)=1/n$. □(Known)

$$\Gamma(S) := \{ t \in V - S \mid \exists s \in S \text{ s.t. } (s,t) \in E \}.$$

- Proof of Lemma 1: If |F|<n, clearly there is no vertex cut Γ(S)⊆F separating 0∈S and 1∈V-S-Γ(S), then assume |F|≥n.
- If there is such a cut Γ(S), then

$$|\Gamma(S)| \geq \frac{|E(S)|}{n} = h(S) \cdot \min\{|S|, |V-S|\} \geq \frac{\min\{|S|, |V-S|\}}{n}.$$

- Hence, if |S|>n|F| and |V-S|>n|F|, then $|\Gamma(S)| > |F|$ and F cannot be such a cut.
- Determining whether |S|>n|F| and |V-S|>n|F| is done by two simple searches (e.g., DFS) from 0 and 1.

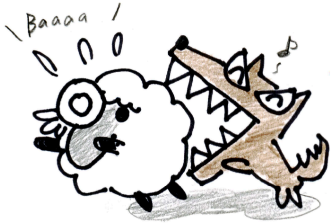


Algorithm

- Then we have the following algorithm:
- **Step 1.** Search (e.g., DFS) from 0
- **If** it reaches 1 **then** output "yes";
- **else if** it finds n|F|+1 vertices
- **then goto** Step 2;
- **Step 2.** Search from 1
- **If** it reaches 0 then output "yes";
- **else if** it finds n|F|+1 vertices
- **then** output "no";
- **end.**

- **Lemma 2**: There is a polynomial time algorithm for SUB-HYPERCUBE CONNECTIVITY.

- **Theorem 4**. If T=∞ and $\mathcal{F}_B$=ϕ, then RIVER CROSSING can be solved in polynomial time.
- Proof: By extending the proof of Lemma 1 (detail omitted.)



Summary

- We give a general formulation RIVER CROSSING.
- If there is no forbidden set on both banks (i.e., $\mathcal{F}_L = \mathcal{F}_R = \phi$),
  - NP hard for any fixed b≥3.
  - P if b≤2. (independent driver model)
- For T=∞, it is in P if $\mathcal{F}_B = \phi$.

- Remaining problem: the case of T=∞ (and $\mathcal{F}_B \neq \phi$).