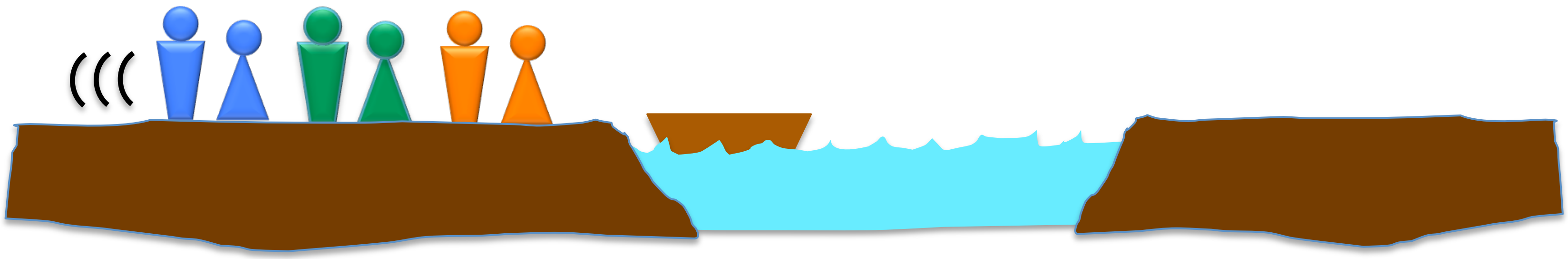
一般化川渡り問題について

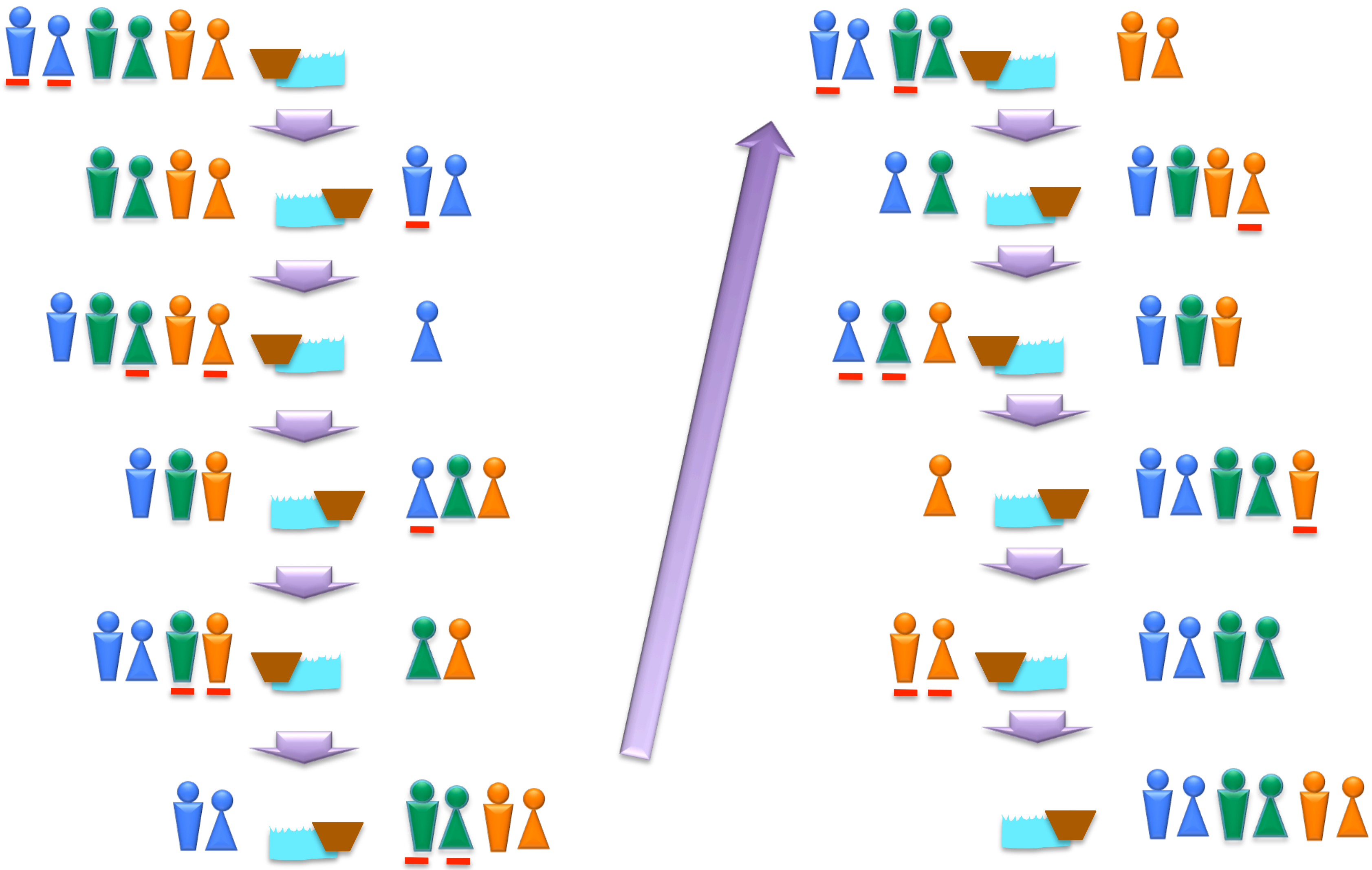
伊藤大雄（京都大学）
Joint work with
Stefan Langerman (Univ. Libre de Bruxelles)
吉田悠一（京都大学）

３人の嫉妬深い男とその妹の問題

- [Alcuin of York,「青年達を鍛えるための諸問題」より]
- ３人の男がそれぞれ一人ずつ未婚の妹を連れて川にさしかかった。
- 二人乗りのボートが一艘あり、それを使って6人が向こう岸に渡りたい。
- ただし、男は警戒心が強く、自分の妹が、自分の居ない所で他の男と同席することは我慢できない。
- 上手く渡る方法を示せ。

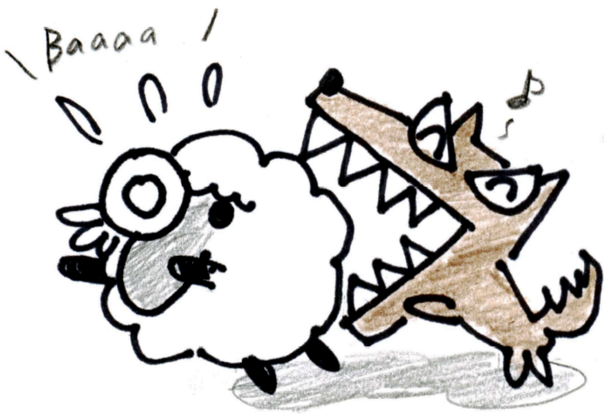


回答



狼と山羊とキャベツの問題

- [Alcuin of York,「青年達を鍛えるための諸問題」より]
- 農夫が狼と山羊とキャベツ（の山）を持って川にさしかかった。
- ボートが一艘あり、それを使って向こう岸に渡りたい。
- ボートには、農夫＋（狼か山羊かキャベツのどれか）しか一度に載せられない。
- 農夫が居ないと「狼は山羊を食べ」「山羊はキャベツを食べ」てしまう。
- 上手く渡る方法を示せ。



- このような問題は「川渡り問題 (River Crossing Problems)」と呼ばれ、多くのバリエーションがある。

We introduce: Generalized River Crossing Problem

PROBLEM: RIVER CROSSING

- INSTANCE:
  - A set of drivers D, A set of customers C, Forbidden sets $\mathcal{F}_L$, $\mathcal{F}_R$, $\mathcal{F}_B \subset 2^{D \cup C}$,
  - The size of the boat $b \in Z^+$, The bound of the # of transportations $T \in Z^+ \cup \{\infty\}$.
- QUESTION: Is there a way to transport all of the drivers and customers from the left bank to the right bank of a river using a boat under the following restrictions?
- RESTRICTIONS:
  1. Initially all drivers, all customers, and the boat are on the left bank.
  2. The capacity of the boat is *b*.
  3. Only drivers can operate the boat.
  4. It is forbidden to transport exactly the members of a forbidden set in $\mathcal{F}_B$ in the boat.
  5. It is forbidden to leave exactly the members of a forbidden set in $\mathcal{F}_L$ (resp., $\mathcal{F}_R$) on the left bank (resp., the right bank).
  6. The number of transportations is at most *T.*

Ex.) the Wolf-Goat-Cabbage Problem

- D={Man}, C={Wolf, Goat, Cabbage}
- $\mathcal{F}_{L} = \mathcal{F}_{R}$ = {{Wolf, Goat}, {Goat, Cabbage},
- {Wolf, Goat, Cabbage}},
- $\mathcal{F}_{B} = \phi$,
- $b = 2$, $T = \infty$.

Note

- $T=\infty \Leftrightarrow T=2^{|P|+1}$ ($P = D \cup C$).
- In this paper, we assume that every driver is allowed to operate the boat alone (Independent-driver model).
- Another formulation by using ALLOWED SETS: This can be reduced to a shortest path problem and can be solved in poly. time.

Past Work

- The allowed set formulation and shortest path approaches [B.R. Schwartz 61], [R. Bellman 62] .
- Unique driver, forbidden pairs (represented by a graph G), forbidden pairs cannot be in the same place without supervision from the unique driver; How large b should be? -> Alcuin # of G [P. Csorba, et al. ESA07] [M. Lampis, V. Mitsou FUN07].

- Our formulation is far more flexible.

Our Results

- Theorem 1: RIVER CROSSING is NP-hard even if $\mathcal{F}_L = \mathcal{F}_R = \phi$ and b=3.
  - Proof: reduced from 3D matching.
- Theorem 2: If $\mathcal{F}_L = \mathcal{F}_R = \phi$ and b=2, then (independent-driver model) RIVER CROSSING can be solved in polynomial time.
  - Proof: easy observations.
- Theorem 3. If $|D|=1$, b=2, $T=\infty$, and $\mathcal{F}_B=\phi$, then RIVER CROSSING can be solved in polynomial time.
  - Proof: Interesting a little. This may extended to wide subclasses of RIVER CROSSING even for b>2.

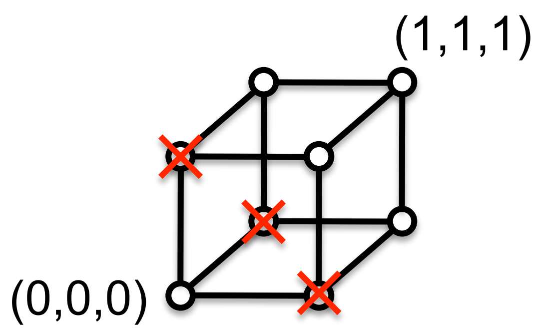
For T=∞ (reachability prob.)

- $H_n=(V_n,E_n)$: the n-dimensional hypercube.

PROBLEM: SUB-HYPERCUBE CONNECTIVITY:

- INSTANCE: A dimension $n \in Z^+$. A set of forbidden vertices $F \subseteq V_n$.
- QUESTION: Is there a path that doesn't use any vertices in F starting from 0 = (0, . . . , 0) and destining to 1 = (1, . . . , 1)?

- Ex.) n=3, F={(0,0,1), (0,1,0), (1,0,0)}
- ⇒Answer "No."



SUB-HYPERCUBE CONNECTIVITY

- Lemma 1: There is a polynomial time algorithm for SUB-HYPERCUBE CONNECTIVITY.

- For a graph G=(V,E), S,T$\subseteq$V,
- E(S,T):={(s,t)$\in$E | s$\in$S, t$\in$T},
- E(S):=E(S,V-S).
- The edge expansion of S of an n-regular graph G is

$$h(S) := \frac{|E(S)|}{n \cdot \min\{|S|, |V-S|\}}.$$

- The edge expansion of G is $h(G) := \min_{S \subseteq V} h(S)$.

- Lemma 2: $h(H_n)=1/n$. □(Known)

$$\Gamma(S) := \left\{ t \in V - S \mid \exists s \in S \text{ s.t. } (s,t) \in E \right\}.$$

- Proof of Lemma 1: If |F|<n, clearly there is no vertex cut Γ(S)⊆F separating 0∈S and 1∈V-S-Γ(S), then assume |F|≥n.
- If there is such a cut Γ(S), then

$$\left|\Gamma(S)\right| \geq \frac{\left|E(S)\right|}{n} = h(S)\cdot \min\left\{\left|S\right|, \left|V-S\right|\right\} \geq \frac{\min\left\{\left|S\right|, \left|V-S\right|\right\}}{n}.$$

- Hence, if |S|>n|F| and |V-S|>n|F|, then there is no such a cut.
- Then we have the following algorithm:

Algorithm

- Step 1. Search (e.g., DFS) from 0
- If it reaches 1 then output “yes”;
- else if it finds n|F|+1 vertices
- then goto Step 2;
- Step 2: Search from 1
- If it reaches 0 then output “yes”;
- elseif it finds n|F|+1 vertices
- then output “no”;
- end.
- □

- Lemma 1: There is a polynomial time algorithm for SUB-HYPERCUBE CONNECTIVITY.

- Theorem 3. If |D|=1, b=2, T=∞, and $\mathcal{F}_B=\phi$, then RIVER CROSSING can be solved in polynomial time.
- Proof: Obtained from Lemma 1 (detail omitted.)

Summary

- We give general formulation RIVER CROSSING.
- If there is no forbidden set on both banks (i.e., $\mathcal{F}_L$ = $\mathcal{F}_R$ =ϕ),
  - NP hard for any fixed b≥3.
  - P if b≤2. (independent driver model)
- For T=∞, P if |D|=1, b=2, and $\mathcal{F}_B$=ϕ.
- This proof uses that the transition graph has a large expansion only! →
- We conjecture that wide subproblems of RIVER CROSSING is in P even for b≥3.