Kaboozle is NP-complete, even in a Strip Form

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Short History:
2010/1/9: At Boston Museum… we met Kaboozle!
2010/2/21 …accepted by 5th International Conference of FUN with Algorithms (FUN 2010)!
Side Story:
"Stretch minimization problem on a strip paper", accepted by
5th International Conference on Origami in Science, Mathematics, and Education (5OSME)

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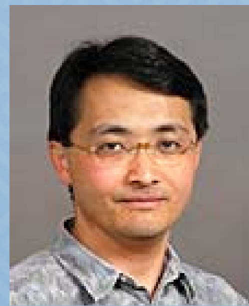

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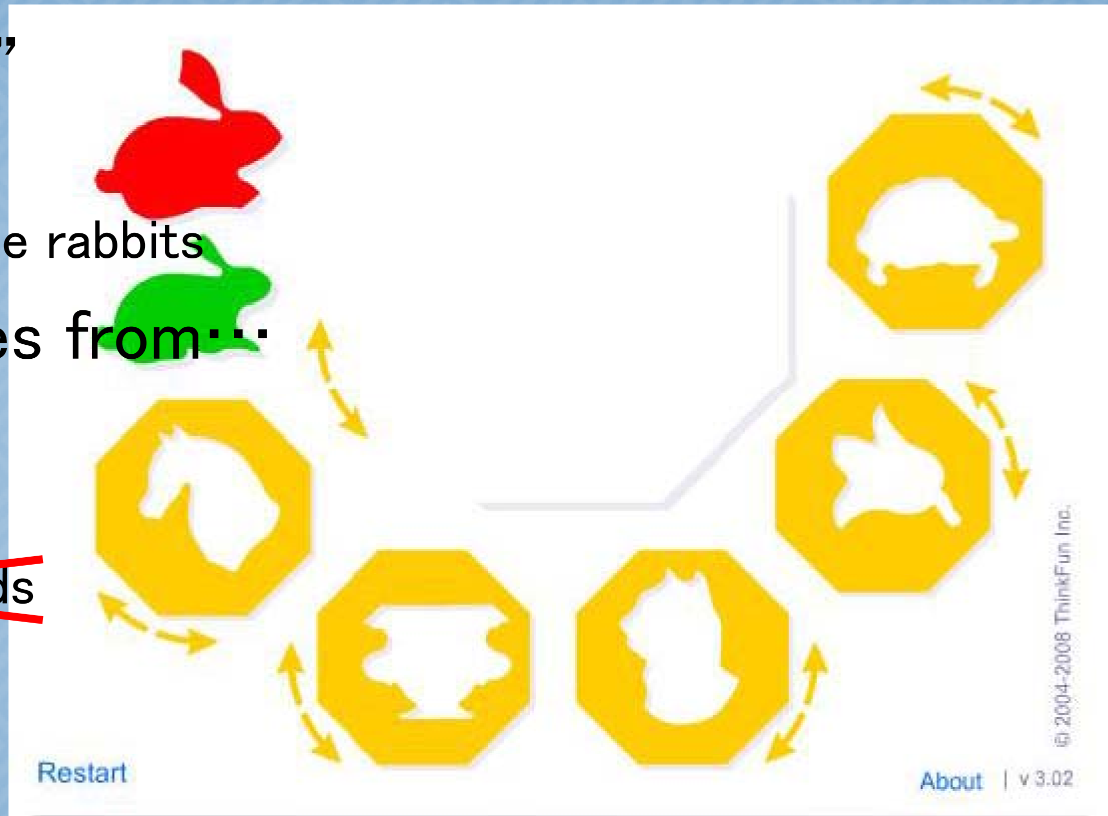
What' s Kaboozle?

- "Labyrinth Puzzle"
  - consists of 4 (square) cards
  - pile them and connect the color path
  - its generalized version seems to be NP-hard.
- It' s Difficulty comes from…
  1. rotation
  2. flipping
  3. ordering of the cards
- Our interest is…
  - the boundary of the difficulty of restricted generalized Kaboozle.

what is the essential of the difficulty?

Related to “Silhouette Puzzle”?

- “Silhouette Puzzle”
  - consists of 5 cards
  - pile them and make the rabbits
- It’s Difficulty comes from…
  1. rotation
  2. flipping
  3. ~~ordering of the cards~~
- Our interest is…



  - the boundary of the difficulty of restricted generalized Kaboozle.

what is the essential of the difficulty?

Very restricted Kaboozle…

- Join them into a strip form like…
  - rotation/flipping are inhibited
  - ordering of the cards are very restricted
  - it seems that "DP from one side works!"…?

Even in this very restricted form,

> Theorem:
> Generalized Kaboozle is still NP-complete
> even in **a strip form**
> with **specified mountain/valley pattern.**

Background about Origami problem

- Any given "mountain-valley pattern" of length n,
  - how many folding ways consistent to the pattern?
  - Uehara showed that it is exponential *on average*!!
- How many folding ways of length n?
  - According to "The On-Line Encyclopedia of Integer Sequences," "The number of folding ways of a strip of n labeled stamps" is obtained up to n=28 by enumeration!
  - These values seem to fit to $\Theta(3.3^n)$
  - Uehara recently obtained the upper/lower bounds of this value; $\Omega(3.07^n)$ and $O(4^n)$,

    which imply that the average value for a random pattern is $\Omega(1.53^n)$ and $O(2^n)$.

Results from the Origami problem

- Observation:
  For a given mountain-valley pattern, the way of folding is unique if and only if the pattern is pleats, that is, "MVMVMV…" .
- Proof:

  (←) Trivial.

  (→) If the pattern contains "MM", we have two choices to pile the paper. Hence it contains neither "MM" nor "VV", which complete the proof.

Results from the Origami problem

- Useful pattern: "shuffle pattern" of length n (n=6):

M V M V M V M V M V **M M** V M V M V M V M V M

- Property:

A shuffle pattern of length n has (exactly) $\binom{2n}{n}$ foldings

Theorem:
Generalized Kaboozle is still NP-complete
even in a strip form
with specified mountain/valley pattern.

- Proof: poly-time reduction from the following NP-complete problem [GJ79]:

1-in-3 3SAT:

Input: $F(x_1,x_2,\ldots,x_n)=c_1 \wedge c_2 \wedge \ldots \wedge c_m$, where

$c_i=(l_i^1 \vee l_i^2 \vee l_i^3)$, $l_i^j = x_k$ or $l_i^j = \neg x_k$

Question: determine if F has an assignment s.t. each clause has exactly one true literal.

Ex: $F(x_1, x_2, x_3, x_4) = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4) \wedge (\neg x_2 \vee x_3 \vee \neg x_4)$

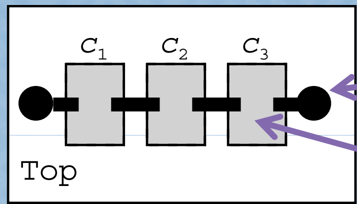
is yes instance with $x_1=1$, $x_2=0$, $x_3=0$, $x_4=1$

Lemma:
Generalized Kaboozle is still NP-complete

- Proof: From the formula, we construct the following Kaboozle cards;

Ex: $F(x_1, x_2, x_3, x_4) = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4) \wedge (\neg x_2 \vee x_3 \vee \neg x_4)$

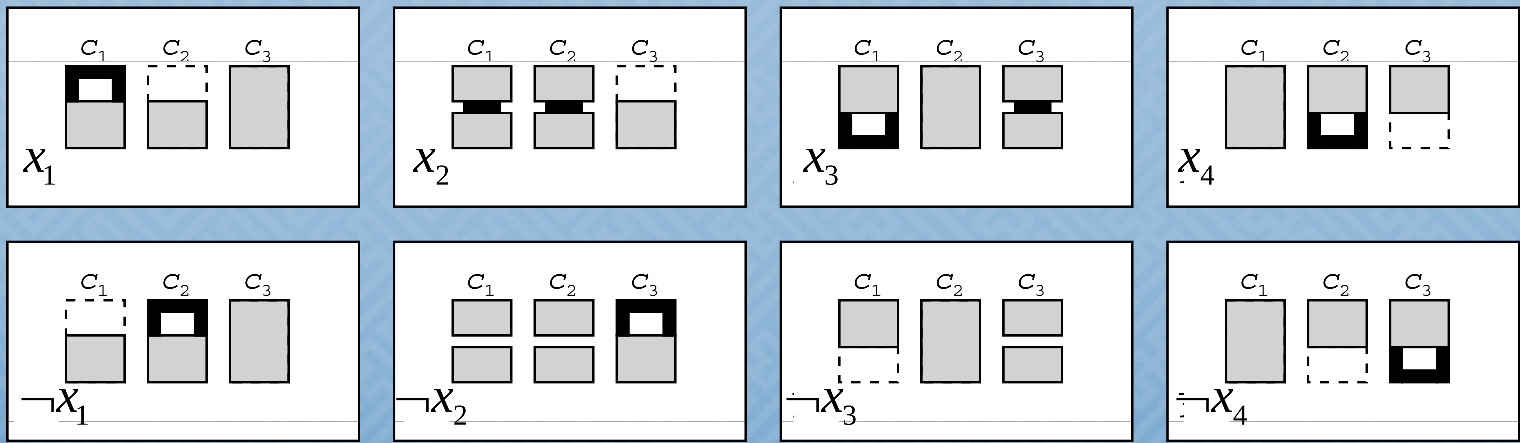
1. top card



the unique path

holes for each clause

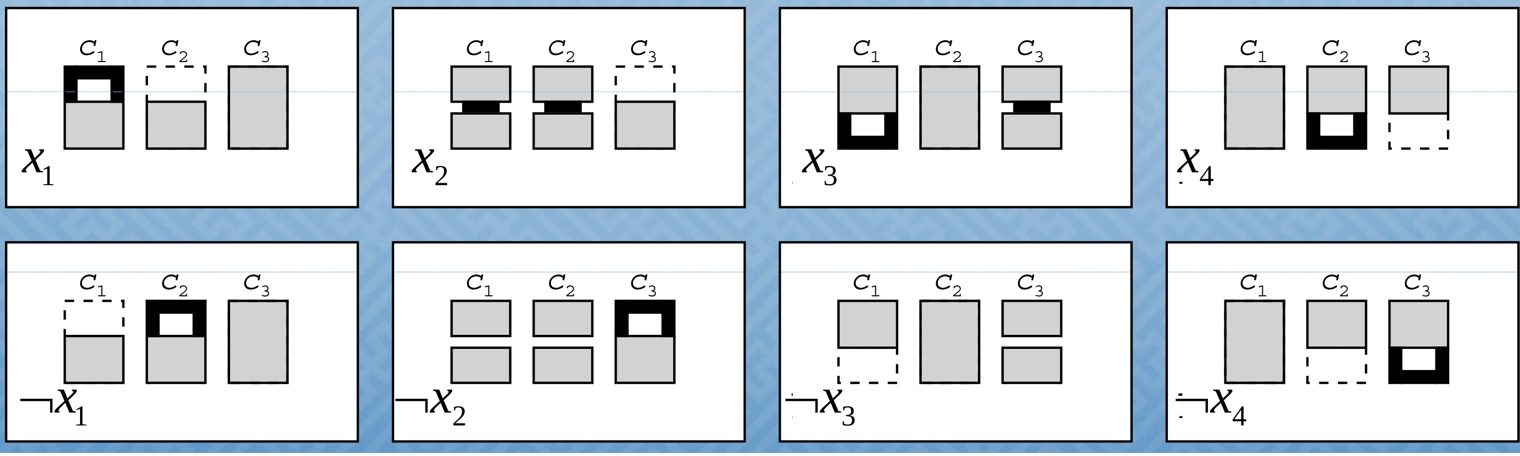
2. variable cards



$F()$ is yes instance with x_1=1, x_2=0, x_3=0, x_4=1,
but fails with x_1=0, x_2=1, x_3=0, x_4=1

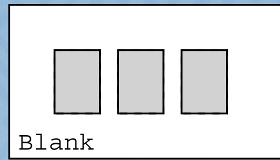
Lemma:
Generalized Kaboozle is still NP-complete

- Proof: From the formula, we construct the following Kaboozle cards (in polynomial time);
  1. top card should be the top (otherwise two endpoints disappear)
  2. for variable cards…
     1. the cards for $\{x_i, \neg x_i\}$ and $\{x_j, \neg x_j\}$ are independent
     2. x_i covers the paths on $\neg x_i$ and vice versa
  3. The set of Kaboozle cards has a solution if and only if the 3SAT formula satisfies the condition.

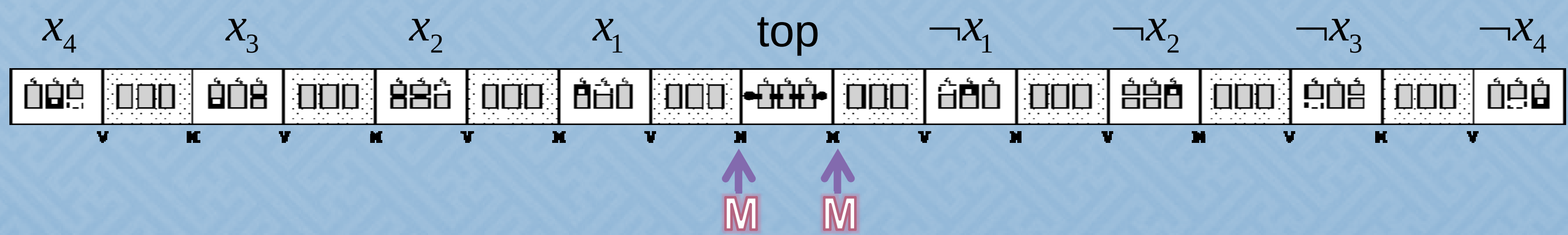


Theorem:
Generalized Kaboozle is still NP-complete
even in a strip form
with specified mountain/valley pattern.

- Proof: poly-time reduction from 1-in-3 3SAT:



We join top cards, variable cards, and $2n$ blank cards in a strip form with the shuffle pattern:

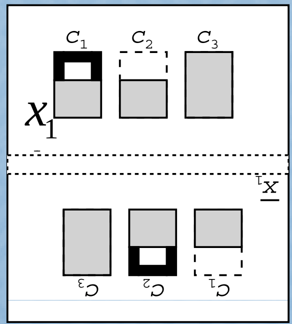


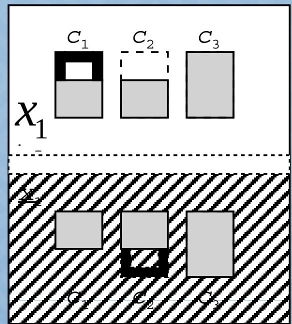
by the lemma and the property of the shuffle pattern, Theorem follows.

Remarks

- "Generalized Kaboozle" is NP-complete even if they are joined in a strip form with/without mountain-valley pattern.
  - So "determine the ordering" is hard enough.
- What happen if ordering of the cards are fixed and
  1. (only) rotation is allowed and/or
  2. (only) flipping is allowed?

  …both are NP-complete.



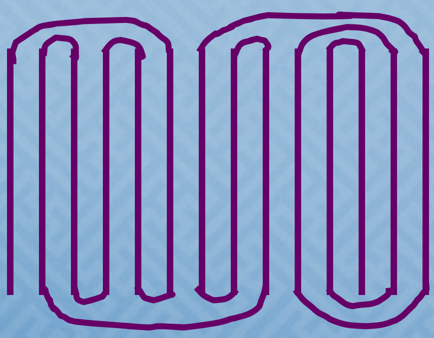


- My personal interest is…
  - For any given mountain-valley pattern, find the "best" folded state, where "best" means that the maximum number of papers between each pair of papers hinged at a crease is minimized.

Appendix

- How many folding ways of length n?
  - Uehara recently obtained the upper/lower bounds of this value; $\Omega(3.07^n)$ and $O(4^n)$.
    - the upper bound $O(4^n)$ comes from the Catalan number.

[Proof] If the paper of length n is folded, the endpoints are nested.



Nest

(()())(()(()))

Combination of $n/2$ pairs of ()= Catalan Number $C_{n/2}$

×

Nest

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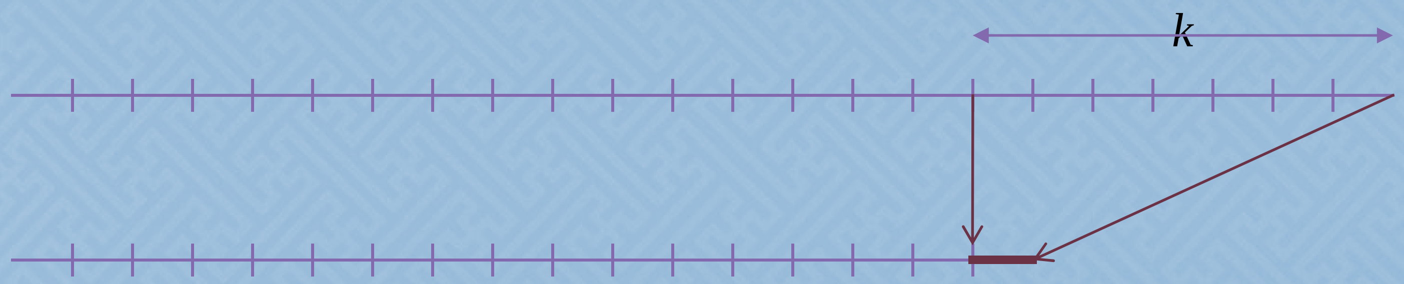
Combination of $n/2$ pairs of ()= Catalan Number $C_{n/2}$

Appendix

- How many folding ways of length n?

[Thm] Its lower bound is $\Omega(3.07^n)$.

[Proof] We consider of folding of the last k unit papers;

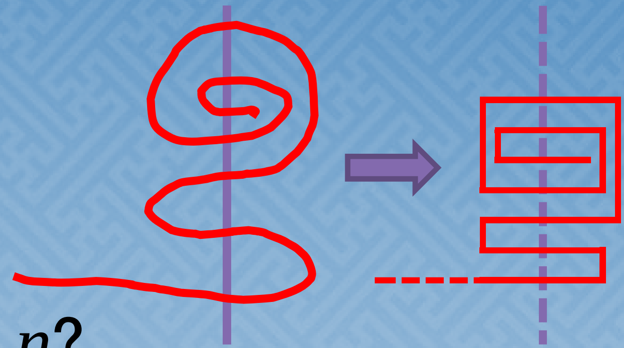


We let

- $f(n)$: the number of folding ways of length n
- $g(k)$: the number of folding ways of length k s.t. the leftmost endpoint is not covered

Then, we have $f(n) \geq (g(k))^{\frac{n}{k-1}} = \left(g(k)^{1/(k-1)}\right)^n$

Appendix



- How many folding ways of length n?

[Thm] Its lower bound is $\Omega(3.07^n)$.

[Proof] We consider of folding of the last k unit papers;

$g(k)$: the number of folding ways of length k s.t.
the leftmost endpoint is not covered
is equal to "the number of ways a semi-infinite directed curve can cross a straight line k times",
A000682 in "The On-Line Encyclopedia of Integer Sequences".
From that site, we have $g(44)=830776205506531894760$.
Thus, by

$$f(n) \geq (g(k))^{\frac{n}{k-1}} = \left(g(k)^{1/(k-1)}\right)^n$$

we have the lower bound.

also obtained
by enumeration